

* When we multiply any matrix by identity matrix, the matrix does not change

$$\text{eg. } \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$$

The transpose of a matrix

$$A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & -3 & -2 & 4 \end{pmatrix} \quad 2 \times 4 \quad A^T = \begin{pmatrix} 2 & 0 \\ 1 & -3 \\ 3 & -2 \\ 5 & 4 \end{pmatrix} \quad 4 \times 2$$

Properties

1- $(A^T)^T = A$

2- $(A+B)^T = A^T + B^T$

3- $(AB)^T = B^T A^T$

4- if $A = A^T \Rightarrow$ Symmetric matrix

if $A = -A^T \Rightarrow$ Skew-Symmetric matrix

Inverse of a matrix

$$AX = B, \quad X = A^{-1}B$$

To find inverse of a matrix:

① Matrix must be square matrix, eg. 2×2 , 3×3 , 4×4 ...

② Not all square matrices have inverse

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}$$

Gauss-Jordan elimination method

Step 1: $(A | I_3)$ Adjoint matrix

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array} \right)$$

Step 2: $(I_3 | A^{-1})$ use row operations to transform matrix A into identity matrix I, the right side of the matrix will be the inverse

$$\begin{array}{l} \xrightarrow{R_2 - R_1} \\ \xrightarrow{R_3 + 6R_1} \end{array} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array} \right) \xrightarrow{\begin{array}{l} R_1 + R_2 \\ R_3 + 4R_2 \end{array}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array} \right)$$

$$\xrightarrow{(-1)R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array} \right) \xrightarrow{\begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \end{array}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}$$

To check: $AA^{-1} = I$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -4 & 3 \end{pmatrix} \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the inverse of Matrix A (if possible)

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{pmatrix} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 + 2R_1}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -7 & 2 & 0 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array} \right)$$

if you get a row of zeros then there is no inverse.

Special Case: Inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ c & a \end{pmatrix}$$

$$\text{Ex } A = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} \quad A^{-1} = \frac{1}{(3 \times 2) - (-1 \times -2)} \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} \quad B^{-1} = \frac{1}{\underset{0}{3 \times 2 - 6}} \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \Rightarrow \text{no inverse}$$

properties of inverse

- 1) $(A^{-1})^{-1} = A$
- 2) $(A^k)^{-1} = (A^{-1})^k \quad (A^4)^{-1} = (A^{-1})^4$
- 3) $(A^T)^{-1} = (A^{-1})^T$
- 4) $(AB)^{-1} = B^{-1}A^{-1}$

$$\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned} \Rightarrow \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

$$AX = B$$

A: coefficient matrix

X: unknown matrix

B: Constant term matrix

$$AX = B$$

$$\underline{A^{-1}} AX = A^{-1} B$$

$$\boxed{X = A^{-1} B}$$

Theorem: If A is an invertible matrix (has inverse) then the system of linear equations $AX = B$ has a unique solution given by $X = A^{-1} B$

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array} \right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array} \right)$$

$$X = A^{-1} B = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

$$x = 2 \quad y = -1 \quad z = -2$$

Elementary matrices

An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single operation

eg

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_2 \times 8} I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine the elementary matrices

① $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Elementary

② $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

not elementary

③ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

not elementary

④ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Elementary

⑤ $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Elementary

⑥ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

not elementary

Find

① $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{pmatrix}$

② $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & 1 \end{pmatrix}$

③ $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & 5 \end{pmatrix}$

When multiplying matrix A with elementary matrix E , the single operation that transformed identity matrix into E is used on A .

Note: The elementary matrix should be on the left side.

The inverse of a matrix

Theorem: If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

Steps to find inverse of elementary matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To get inverse of E reverse the operation used on I to transform it to E and use it on I .

$$I \xrightarrow{qR_3} E$$

$$\leftarrow I \xrightarrow{\frac{1}{q}R_3} E^{-1}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{q} \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 + 5R_3$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 - 5R_3$$

Operation	Inverse operation
1. Interchange rows p, q	Interchange rows p, q
2. multiply row p by $c \neq 0$	multiply row p by $\frac{1}{c}$
3. Add k times row p to row q	Subtract k times row p from row q